DRP SUMMER 2025: THE SHERRINGTON-KIRKPATRICK SPIN GLASS MODEL

Myles Allred

Mentored by: Jennifer Vaccaro

What is a spin glass?

A system with many interacting components that has the following features

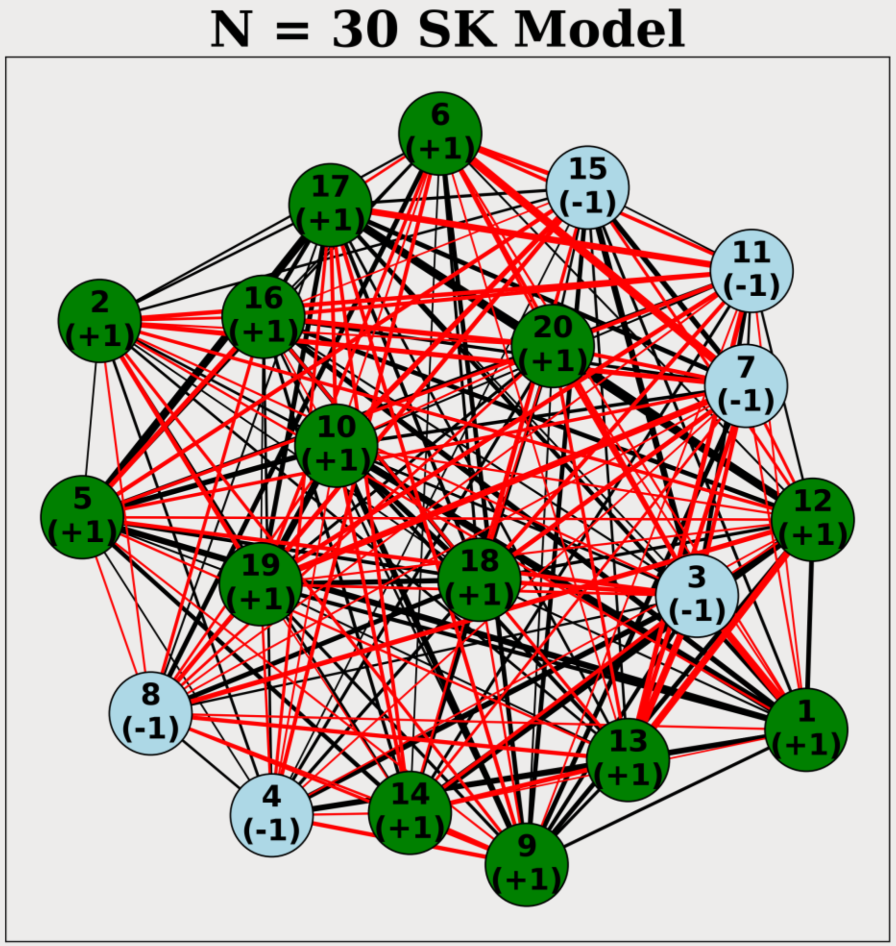
1.) **Quenched Disorder :**

- Fixed variables that do not change in the Hamiltonian as system evolves (e.g. the coupling matrix J)

2.) **Frustration :**

- System is unable to satisfy all local constraints simultaneously

The Sherrington-Kirkpatrick Mean-Field Model



Black = Positive coupling
Red = Negative coupling

Called a *mean field model*: every node interacts with some mean field of the rest of the system, rather than have distinct local/sparse interactions

Fully connected graph with **N=30** degrees of freedom. Each node $\boldsymbol{i}$ has state $\boldsymbol{s_i} = \pm\mathbf{1}$.

$$J = \begin{bmatrix} 0 & 0.015 & -0.687 & 0.0139 \\ 0.0159 & 0 & 0.091 & -0.017 \\ -0.687 & 0.091 & 0 & 0.058 \\ 0.019 & 0.016 & -0.158 & 0 \end{bmatrix}$$

The strength of each coupling $\boldsymbol{J_{ij}}$ is independently chosen from a normal distribution.

System Measures

Hamiltonian: A cost function of a certain configuration of spins $S = [s_1, \dots, s_N]$, tells us how satisfied a model is in that configuration.

Magnetization: How aligned the spins are. Completely aligned (ferromagnetic) means high magnetization value.

Boltzmann Probability: Probability of the system being in a certain configuration based on its energy.

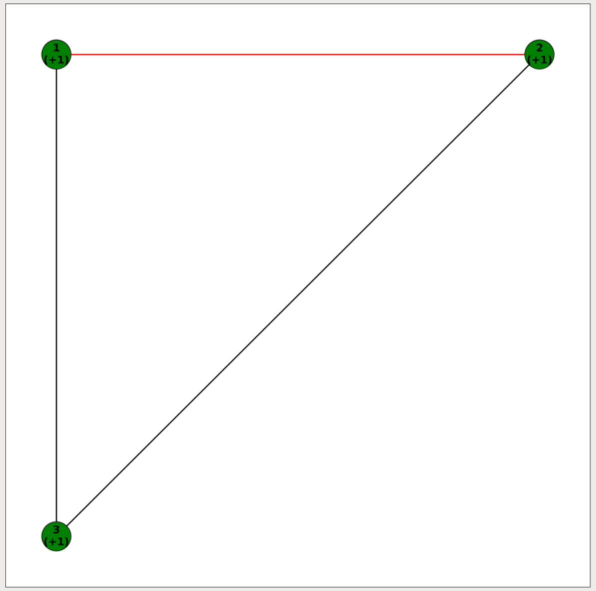
Partition Function (sum): Normalizing factor for the Boltzmann probability so that the probabilities of all possible configurations sum to 1.

Free Energy: An energy measure of the entire system, over all its possible spin configurations.

The Hamiltonian - A cost function

$$\mathcal{H}(S, J) = -\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j - h\sum_{i}\sigma_i$$

$$\text{where } \sigma_i = \pm 1$$



$$N = 3$$

$$J = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$S_A = [1, 1, 1]$$

$$\mathcal{H}(S_A, J) = -\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j$$

$$= -(J_{12}\sigma_1\sigma_2 + J_{13}\sigma_1\sigma_3 + J_{23}\sigma_2\sigma_3)$$

$$= -(1(1)(1) + 1(1)(1) + -1(1)(1))$$

$$= -1$$

The Boltzmann Distribution:

Probability of being in a certain configuration at temperature T

$$\mathbb{P}^{Boltzmann}(S_i, J, \beta) = \frac{1}{Z(J, \beta)} e^{-\beta H(S_i, J)}$$

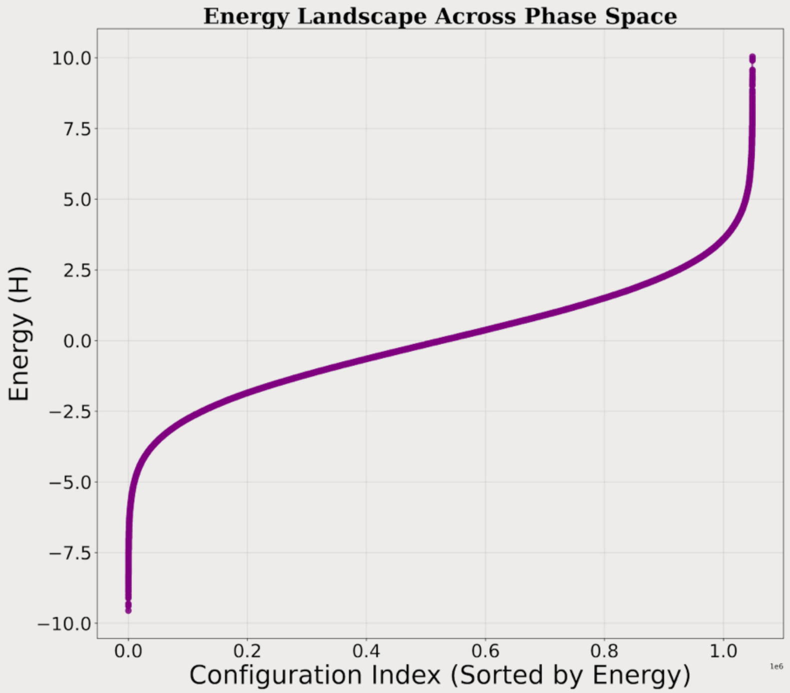
$$\text{where } \beta = 1/T$$

Partition Function (normalizing factor):

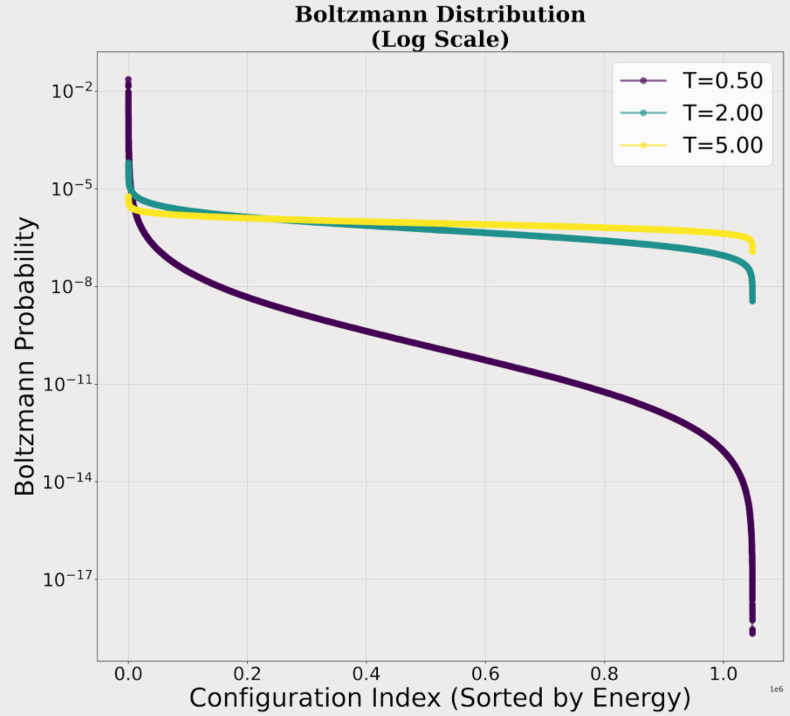
$$Z(J, \beta) = \sum_{S} e^{-\beta H(S, J)}$$

Ways to Analyze the System:

Approach	Method	Pros	Cons
1. Brute force	Compute all configurations and Hamiltonians	Exact, conceptually simple	Computationally intractable, doesn't capture probability of states



Energy Landscape Across Phase Space
Energy (H)
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0.0
0.2
0.4
0.6
0.8
1.0
1e6
Configuration Index (Sorted by Energy)



Boltzmann Distribution
(Log Scale)
T=0.50
T=2.00
T=5.00
Boltzmann Probability
10^{-2}
10^{-5}
10^{-8}
10^{-11}
10^{-14}
10^{-17}
0.0
0.2
0.4
0.6
0.8
1.0
1e6
Configuration Index (Sorted by Energy)

Ways to Analyze the System:

Approach	Method	Pros	Cons
1. Brute force	Compute all configurations and Hamiltonians	Exact, conceptually simple	Computationally intractable, doesn't capture probability of states
2. Monte Carlo Sampling with Metropolis Algorithm	Random walk based on probability of increasing energy	Efficient and simple, represents behavior at thermodynamic equilibrium	Only tells us about a single walk, may get stuck in local energy minima.

Metropolis-Hastings Algorithm

Approximate a models' Boltzmann distribution without computing the partition function directly.

The Algorithm:

1.) Initialize the model in a random configuration

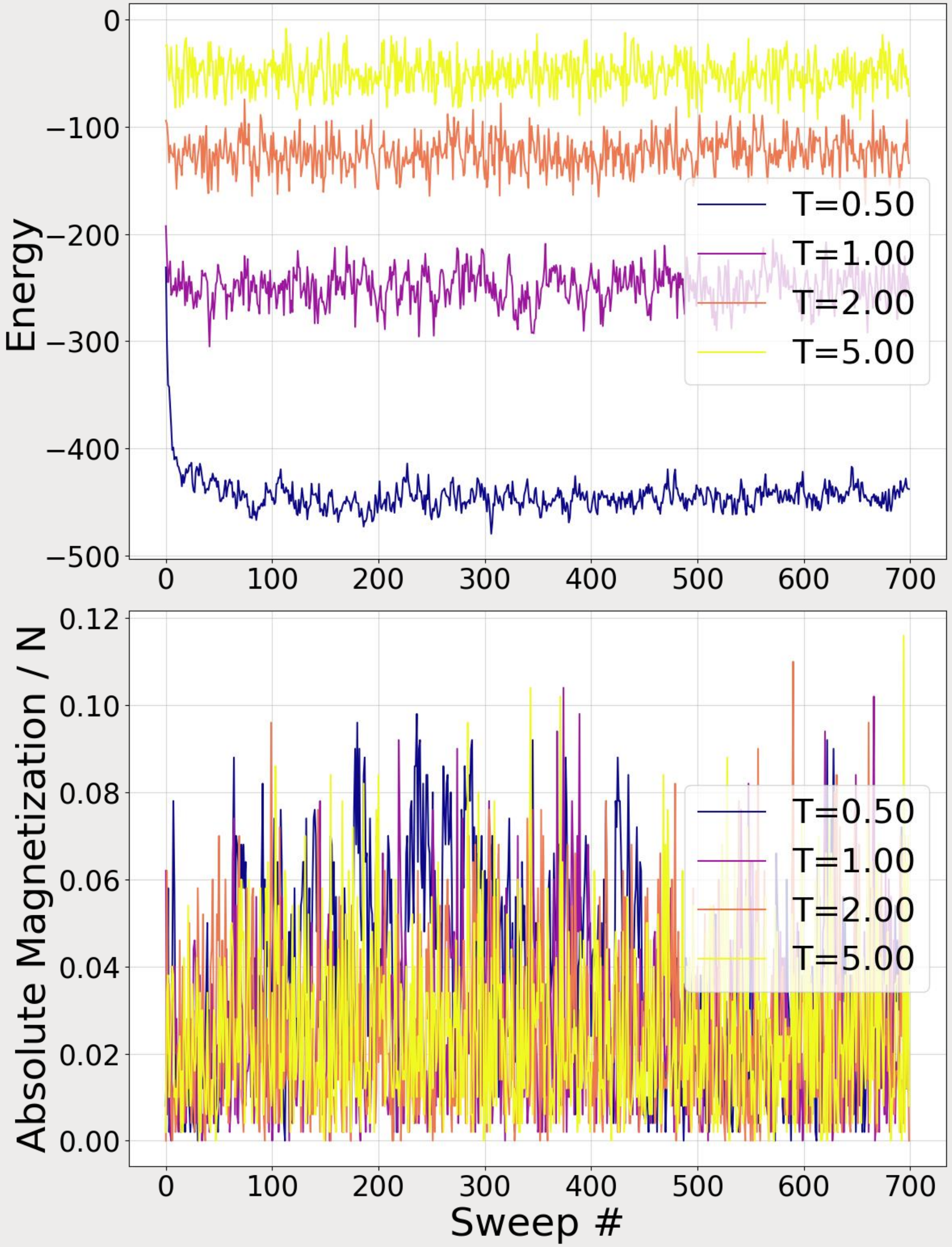
2.) Propose flipping one spin and calculate $\Delta H = \Delta_{new} - \Delta_{old}$

3.) Choose to accept or reject the move:

- $\Delta \mathrm{H} \leq 0$: always accept it
- $\Delta \mathrm{H} > 0$: accept with $\mathrm{P(accept)} = \mathrm{e}^{-\beta \Delta \mathrm{H}}$

4.) Take many of these steps

Metropolis Evolution for a N=1000 SK Model



Ways to Analyze the System:

Approach	Method	Pros	Cons
1. Brute force	Compute all configurations and Hamiltonians	Exact, conceptually simple	Computationally intractable, doesn’t capture probability of states
2. Monte Carlo Sampling with Metropolis Algorithm	Random walk based on probability of increasing energy	Efficient and simple, represents behavior at thermodynamic equilibrium	Only tells us about a single walk, may get stuck in local energy minima.
3. Replica Method	Trick to find analytic solution	Exact solution that tells us about all equilibrium configurations	Difficult to solve, requires an ansatz (assumption) about the overlap of equilibrium configurations.

Replica Method

The free energy is self averaging, meaning that if we compute its average over the set of all possible couplings, we understand it for a typical realization of J.

$$\langle\langle F(J)\rangle\rangle_{\mathscr{J}} = \langle\langle -T \ln Z(J)\rangle\rangle_{\mathscr{J}}$$

To avoid this difficult average, we exploit the identity

$$\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n} = \lim_{n\to 0} \frac{\partial}{\partial n} Z^n$$

Then it is possible to average over an integer power of $Z(J, \beta)$ and take the $n \to 0$ limit.

Remember:

$$Z(J,\beta) = \sum_S e^{-\beta H(S,J)} = \sum_S e^{\beta \sum_{i<j} J_{ij}\sigma_i\sigma_j}$$

We introduce n identical but distinct configurations S^a for $a = 1, \dots, n$ and multiply their partition functions:

$$Z^n = \left(\sum_{\{S^1\}} e^{\beta \sum_{i<j} J_{ij}\sigma_i^1\sigma_j^1}\right) \times \left(\sum_{\{S^2\}} e^{\beta \sum_{i<j} J_{ij}\sigma_i^2\sigma_j^2}\right) \times \cdots \times \left(\sum_{\{S^n\}} e^{\beta \sum_{i<j} J_{ij}\sigma_i^n\sigma_j^n}\right)$$

And take the average of that:

$$\langle\langle Z^n \rangle\rangle_{\mathscr{J}} = \left\langle\left\langle \sum_{\{S^1,\dots,S^n\}} e^{\beta \sum_{a=1}^n \sum_{i<j} J_{ij}\sigma_i^a\sigma_j^a} \right\rangle\right\rangle_{\mathscr{J}}$$

This average can be performed because it can be reduced to Gaussian integrals with the identity

$$\langle e^{zx} \rangle_z = e^{1/2\sigma^2 x^2}$$

Where z is a zero mean Gaussian with variance σ^2

References

Advani, M., Lahiri, S., Ganguli, S., 2013. Statistical mechanics of complex neural systems and high dimensional data. J. Stat. Mech. 2013, P03014. https://doi.org/10.1088/1742-5468/2013/03/P03014

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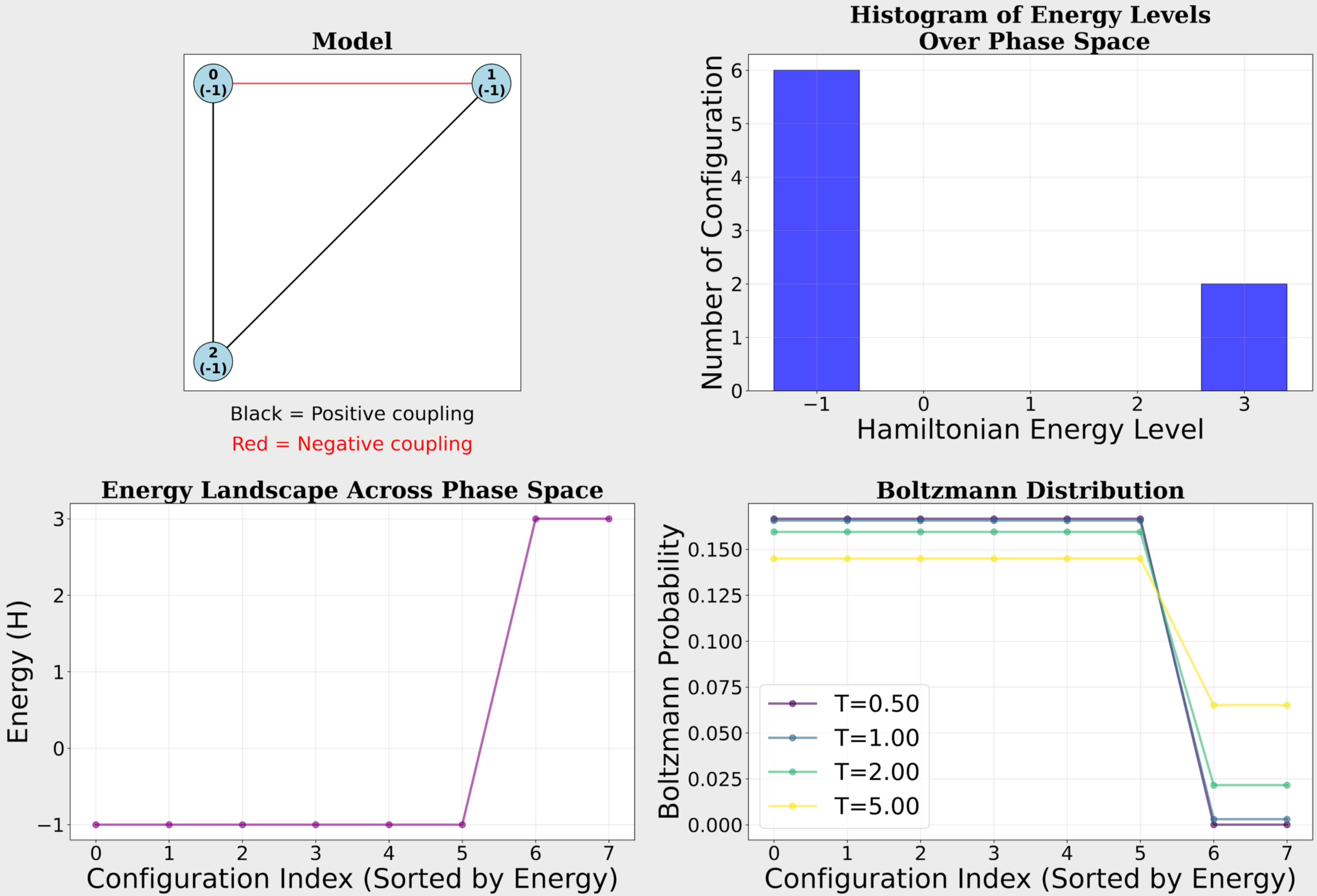
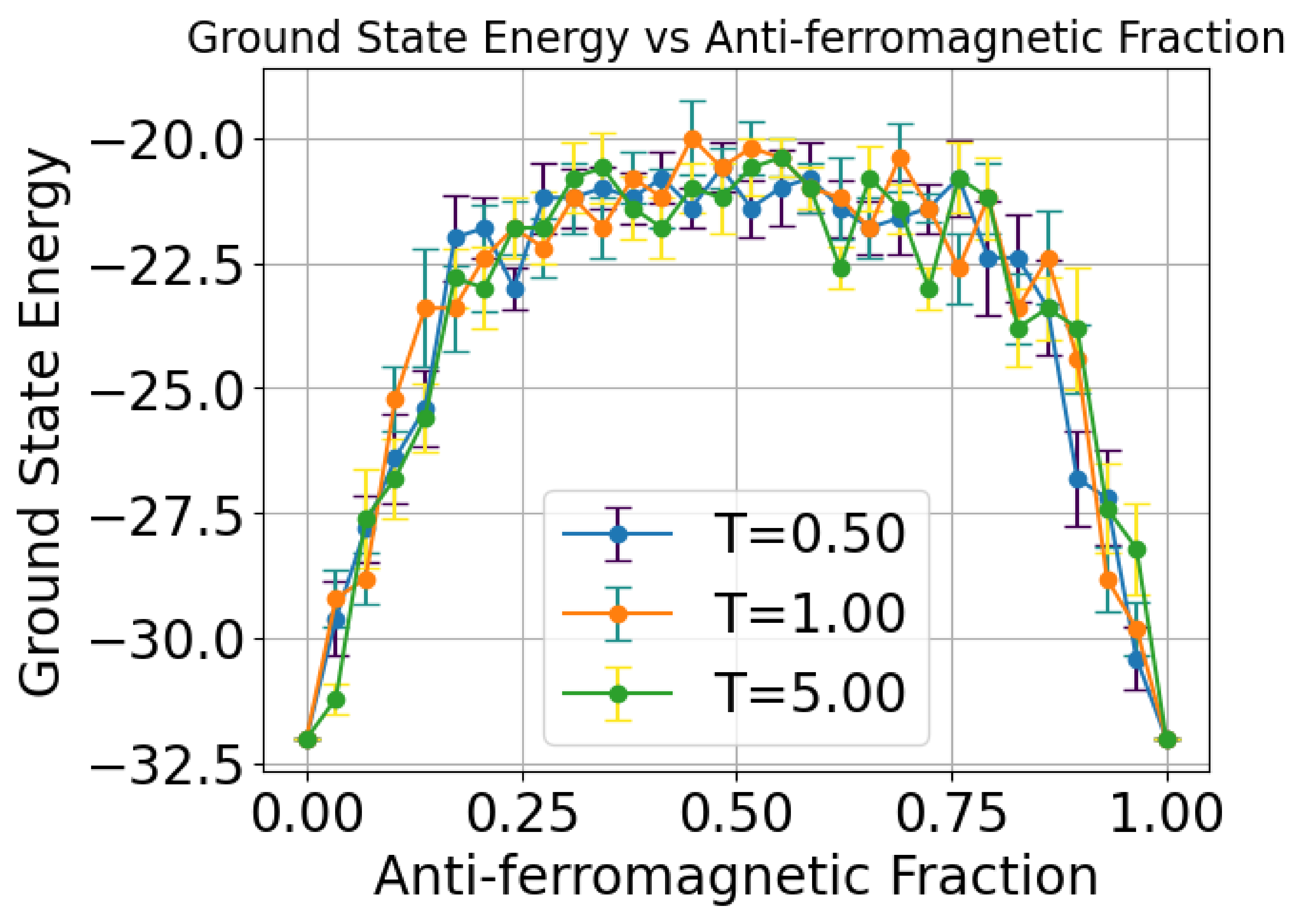
The Ising Model [WWW Document], n.d. URL https://stanford.edu/~jeffjar/statmech/intro4.html (accessed 6.30.25).

Extra graphics:

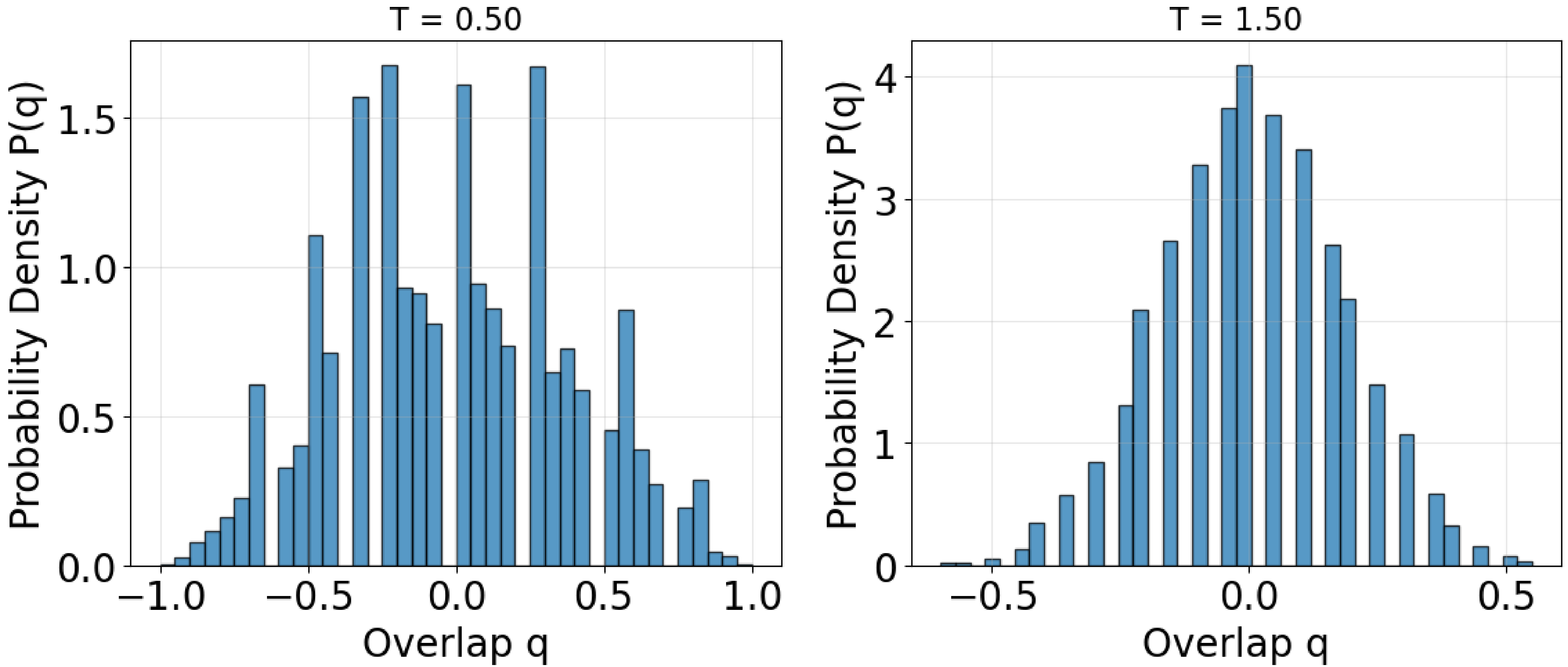
Frustrated Triangle



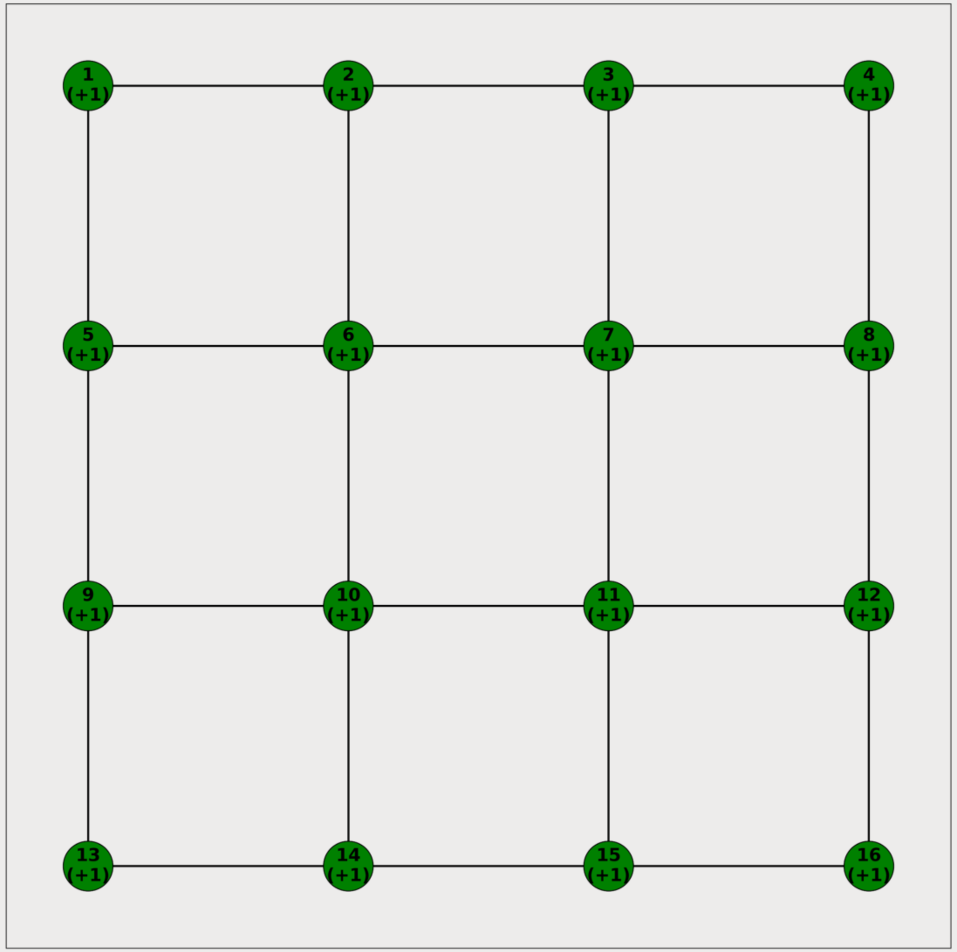


Ground State Energy vs Anti-ferromagnetic Fraction
Ground State Energy
Anti-ferromagnetic Fraction
−20.0
−22.5
−25.0
−27.5
−30.0
−32.5
0.00
0.25
0.50
0.75
1.00
T=0.50
T=1.00
T=5.00

Edwards-Anderson Overlap Distribution P(q) for N=40 SK Model

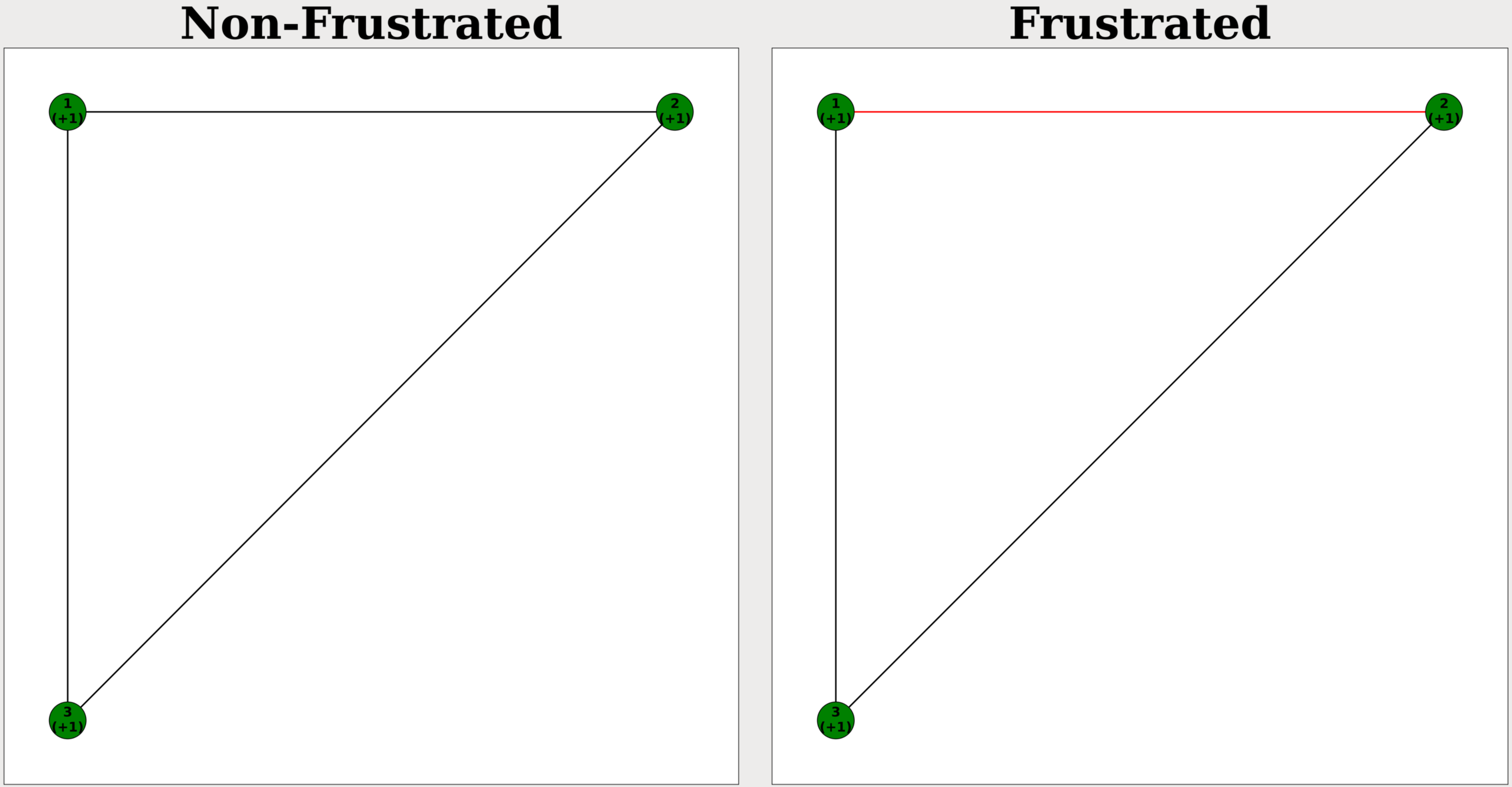


Statistical Mechanics



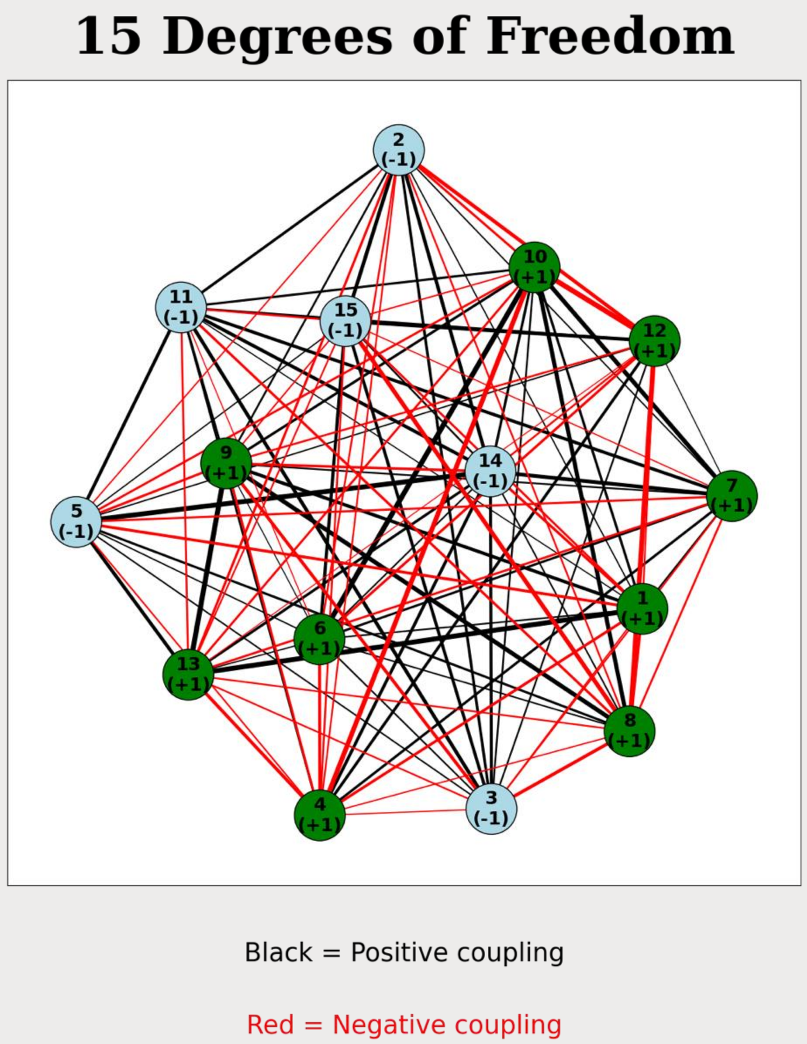
Using statistical methods and probability theory to understand the macroscopic properties of systems made up of many interacting microscopic components (degrees of freedom).

For Ising models, the state of a component, or its “spin” is +1 or -1.



Non-Frustrated
Frustrated
1 (+1)
2 (+1)
3 (+1)
1 (+1)
2 (+1)
3 (+1)

The Sherrington-Kirkpatrick Model:

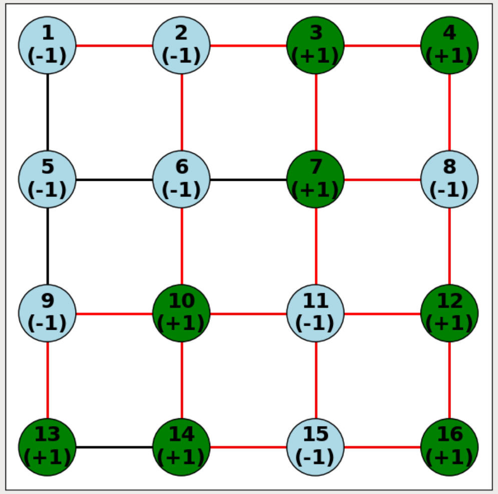


A mean field model where each J_{ij} is independently chosen from a zero mean Gaussian with variance 1/N

This means the couplings are randomly positive and negative, and there is therefore a great deal of frustration in this model.

Random Frustrated Lattice

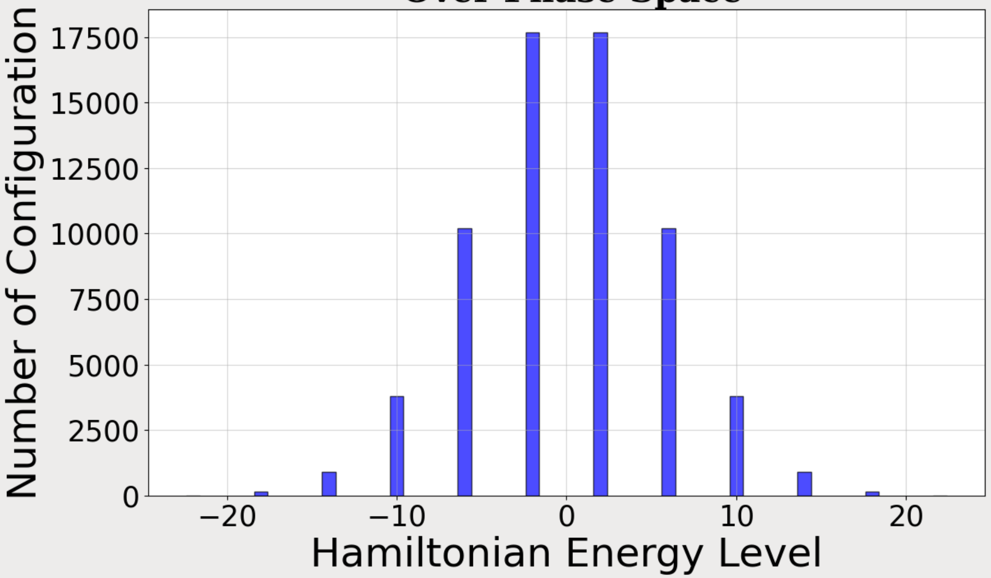
Model



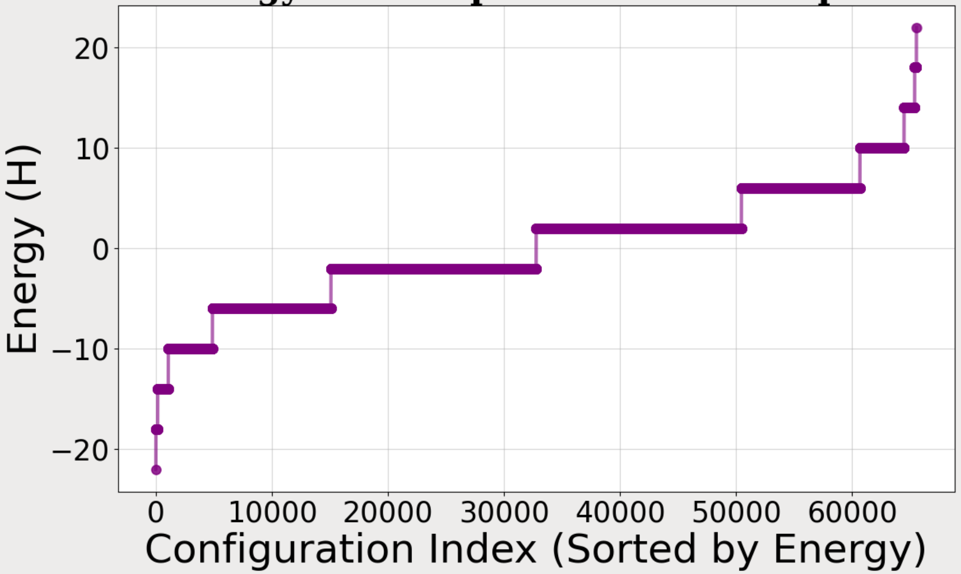
Black = Positive coupling

Red = Negative coupling

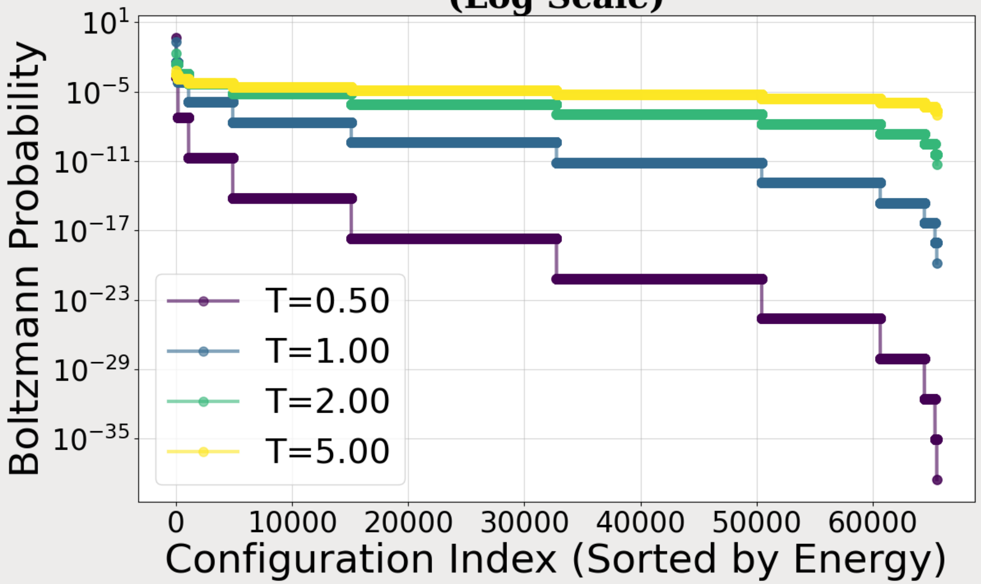
Histogram of Energy Levels Over Phase Space



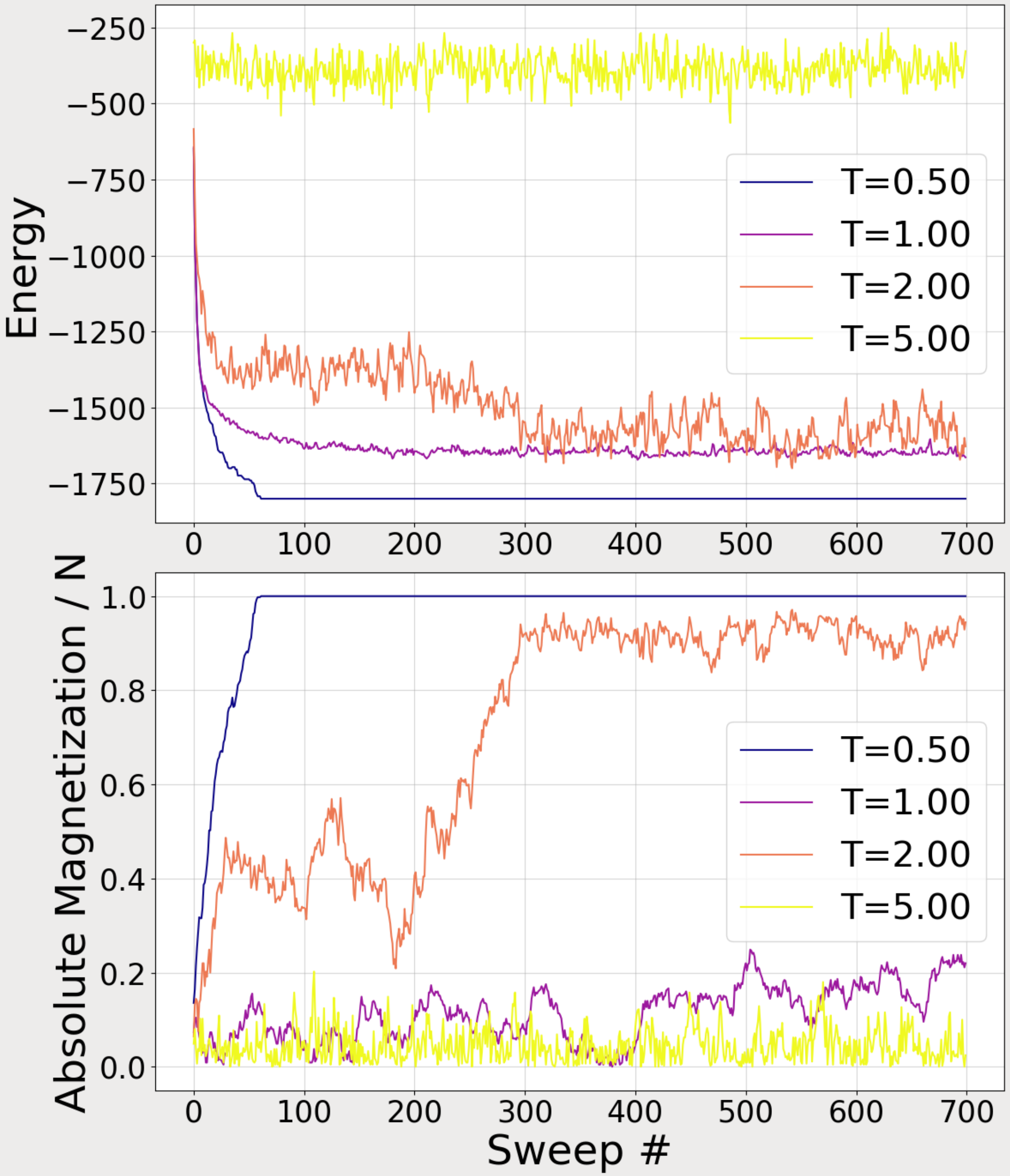
Energy Landscape Across Phase Space



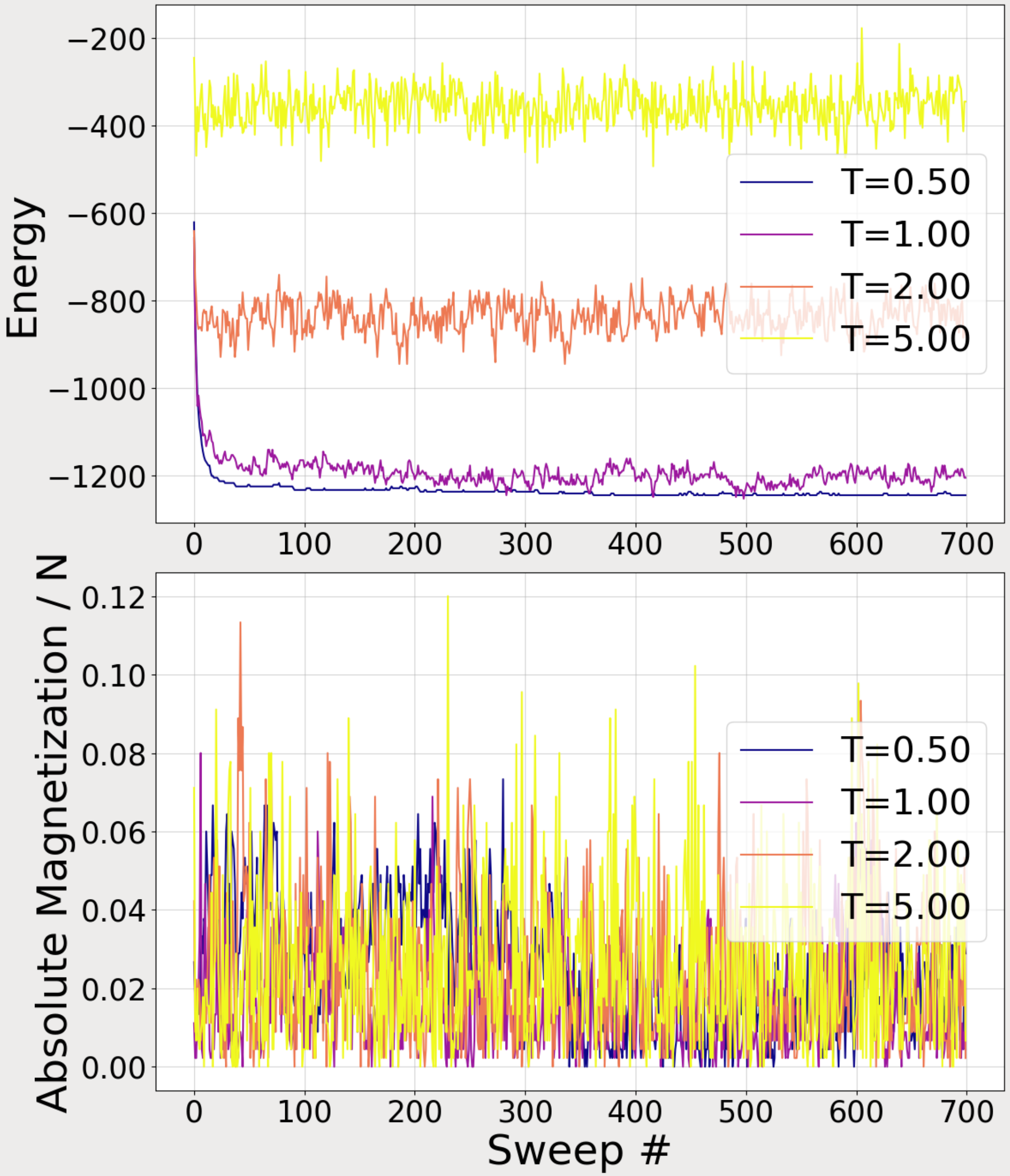
Boltzmann Distribution (Log Scale)



Metropolis Evolution for a Non-Frustrated 30x30 Lattice



Metropolis Evolution for a Frustrated 30x30 Lattice

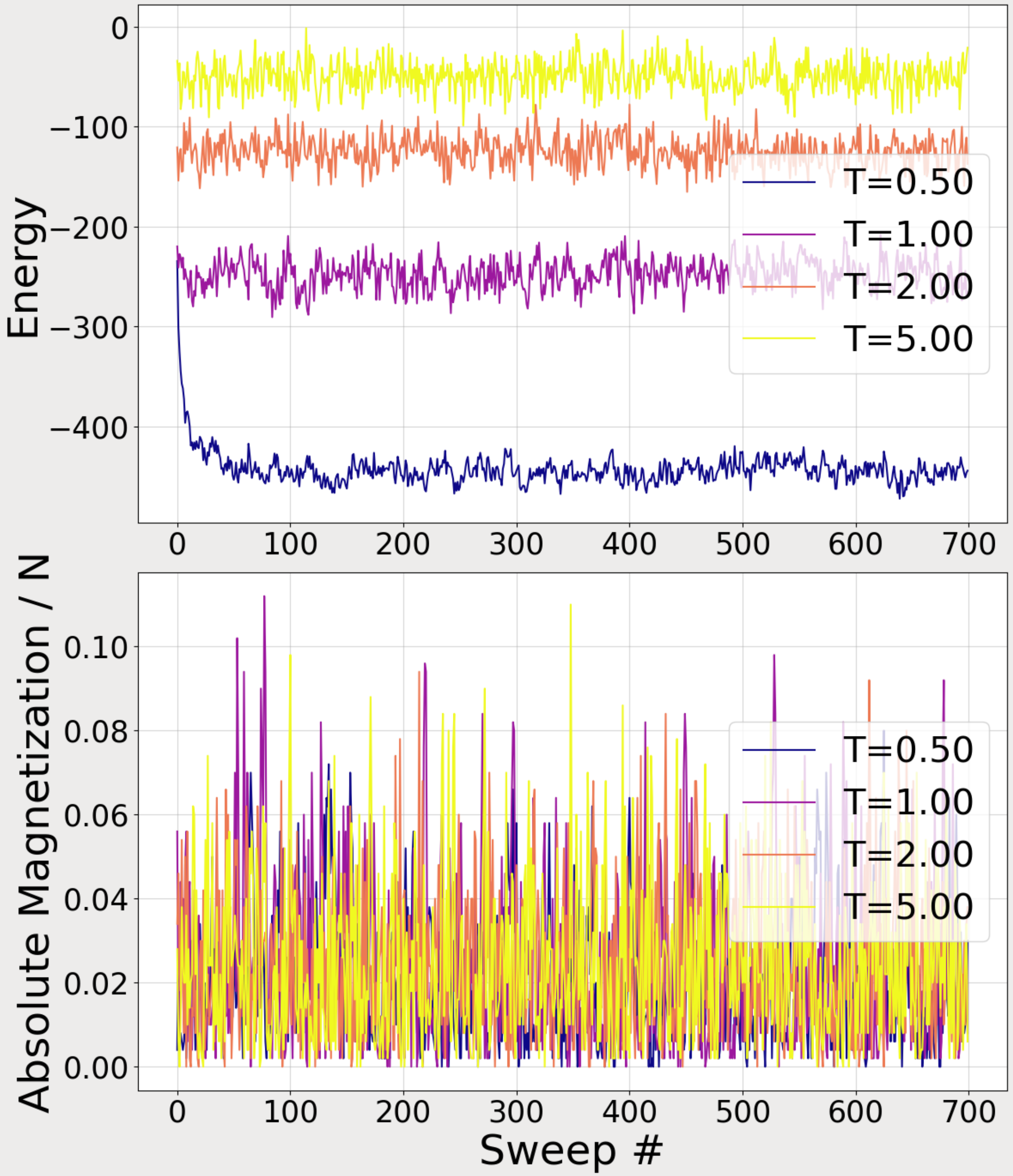


Mean Field Approximation:

Assume every particle interacts with every other particle in the system instead of keeping track of localized interactions.

In other words, each particle interacts with some "mean field" of the entire system.

Metropolis Evolution for a N=1000 SK Model



Unfrustrated Triangle